MATH 4010 Quiz 2 Solution.

1.) Consider the quadratic form $f(x, y, z) = -3x^2 - 2y^2 - 4z^2 + 2xy - 2xz + 4yz$.

   (a) (1 pts) Express f in the form of $\begin{bmatrix} x & y & z \end{bmatrix} A \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ where A is a real-valued symmetric 3×3 matrix.

   Find A..

   Solution: $A = \begin{bmatrix} -3 & 1 & -1 \\ 1 & -2 & 2 \\ -1 & 2 & -4 \end{bmatrix}$

   (b) (2 pts) Use Sylvesters criterion to determine the definiteness of A, and thus f.
   Grading scheme: Take off one point off if one of these steps is missing.
   Solution: Grading scheme: Take off one point off if one of these steps is missing.

   We compute leading principle minors first. $a_{11} = -3 < 0$, $det \begin{bmatrix} -3 & 1 \\ 1 & -2 \end{bmatrix} = 6 - 1 = 5 > 0$ and $det \begin{bmatrix} -3 & 1 & -1 \\ 1 & -2 & 2 \\ -1 & 2 & -4 \end{bmatrix} = -24 - 2 - 2 + 2 + 4 + 12 = -10 < 0$. So A is negative definite.

2. Let $A = \begin{bmatrix} -8 & 4 & 4 \\ 4 & 1 & -5 \\ 4 & -5 & 1 \end{bmatrix}$.

   (a) (4 pts) Find the eigenvalues of A and their corresponding eigenvectors if you are given the fact that $det(A - \lambda I_3) = -\lambda^3 - 6\lambda^2 + 72\lambda$.
   Grading scheme: Total 4 points, One point in determine eigenvalues, one point in determining each eigenvectors.
   Solution:
   Solving $det(A - \lambda I_3) = -\lambda^3 - 6\lambda^2 + 72\lambda = -\lambda(\lambda^2 + 6\lambda - 72) = -\lambda(\lambda - 6)(\lambda + 12) = 0$, we have $\lambda = 0$, $\lambda = 6$ or $\lambda = -12$. So the eigenvalues are 0, 6, -12.
   $\lambda = 0$,

$$A - \lambda I_3 = \begin{bmatrix} -8 & 4 & 4 \\ 4 & 1 & -5 \\ 4 & -5 & 1 \end{bmatrix} \sim \begin{bmatrix} -2 & 1 & 1 \\ 4 & 1 & -5 \\ 4 & -5 & 1 \end{bmatrix} (row1 := (1/2) \cdot row1)$$

$$\sim \begin{bmatrix} -2 & 1 & 1 \\ 0 & 3 & -3 \\ 0 & -3 & 3 \end{bmatrix} (row2 := -2 \cdot row1 + row2, row3 := -2 \cdot row1 + row3)$$

$$\sim \begin{bmatrix} -2 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} (row2 := (1/3)row2, row3 := row2 + row3) \sim \begin{bmatrix} -2 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} (row1 := -row2 + row1)$$

$$\sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} (row1 := (-1/2)row1)$$

   Let $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Then v is an eigenvector with eigenvalue 0 if $x - z = 0$ and $y - z = 0$. So

$v = \begin{bmatrix} z \\ z \\ z \end{bmatrix} = z \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Thus $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector with eigenvalue 0.

$\lambda = 6$,

$$A - \lambda I_3 = \begin{bmatrix} -8-6 & 4 & 4 \\ 4 & 1-6 & -5 \\ 4 & -5 & 1-6 \end{bmatrix} = \begin{bmatrix} -14 & 4 & 4 \\ 4 & -5 & -5 \\ 4 & -5 & -5 \end{bmatrix} \sim \begin{bmatrix} -14 & 4 & 4 \\ 4 & -5 & -5 \\ 0 & 0 & 0 \end{bmatrix} (row3 := -row2 + row3,)$$

$$\sim \begin{bmatrix} 0 & 27 & 27 \\ 4 & -5 & -5 \\ 0 & 0 & 0 \end{bmatrix} (row1 := -2 \cdot row1 + 7row2)$$

$$\sim \begin{bmatrix} 0 & 1 & 1 \\ 4 & -5 & -5 \\ 0 & 0 & 0 \end{bmatrix} (row1 := (1/27)row1) \sim \begin{bmatrix} 4 & -5 & -5 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} (row1 <-> row2)$$

$$\sim \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} (row1 := 5row2 + row1)$$

Let $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Then v is an eigenvector with eigenvalue 0 if $x = 0$ and $y + z = 0$. So $v = \begin{bmatrix} 0 \\ -z \\ z \end{bmatrix} = z \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$. Thus $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ is an eigenvector with eigenvalue 6.

$\lambda = -12$,

$$A - \lambda I_3 = \begin{bmatrix} -8+12 & 4 & 4 \\ 4 & 1+12 & -5 \\ 4 & -5 & 1+12 \end{bmatrix} = \begin{bmatrix} 4 & 4 & 4 \\ 4 & 13 & -5 \\ 4 & -5 & 13 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 4 & 13 & -5 \\ 4 & -5 & 13 \end{bmatrix} (row1 := 1/4row1)$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 9 & -9 \\ 0 & -9 & 9 \end{bmatrix} (row2 := -4 \cdot row1 + row2, row3 := -4 \cdot row1 + row2)$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} (row2 := (1/9)row2, row3 := row3 + row2) \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} (row1 := -row2 + row1)$$

Let $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Then v is an eigenvector with eigenvalue 0 if $x + 2z = 0$ and $y - z = 0$. So $v = \begin{bmatrix} -2z \\ z \\ z \end{bmatrix} = z \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$. Thus $\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector with eigenvalue -12.

(b) (2 pts) Diagonalize A. That is, find a orthogonal matrix P and a diagonal matrix D such that $P^T AP = D$.
Grading scheme: One point in determining P and one point in determining D.
Solution: Let $V_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $V_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ and $V_3 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$. Then $\frac{V_1}{||V_1||} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ 1\frac{1}{\sqrt{3}} \end{bmatrix}$, $\frac{V_2}{||V_2||} = \begin{bmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$,

and $\frac{V_3}{||V_3||} = \begin{bmatrix} \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$ is an orthonormal basis of eigenvectors with eigenvalues $0, 6, -12$. Let $P = \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}$ and $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & -12 \end{bmatrix}$. Then $P^T AP = D$.

3. It is required to maximize $f(x, y) = 1 - 2xy$ subject to $x + y \geq 1$, $x^2 + y^2 = 1$.

   (a) (1 pts) Write down the classical Lagrangian for this problem, $L(x, y, \lambda, \mu)$, where λ, μ are the Lagrange multipliers.
   Grading scheme: One point
   Solution: This is a maximization problem. We write the constraint as $-x - y + 1 \leq 0$ and $x^2 + y^2 = 1$.
   Let $L(x, y, \lambda, \mu) = 1 - 2xy - \lambda(1 - x - y) - \mu(x^2 + y^2 - 1)$.

   (b) (2 pts) State the four equality and two inequality conditions required to determine the maximizers.
   Grading scheme: Each part 1/6 point.
   Solution: We have $\frac{\partial L}{\partial x} = -2y+\lambda-2\mu x = 0$, $\frac{\partial L}{\partial y} = -2x+\lambda-2\mu y = 0$, $\lambda(x+y-1) = 0$, $x^2+y^2 = 1$, $\lambda \geq 0$ and $x + y \geq 1$.

   (c) (5 pts) Find the maximizer and the global maximum of this problem. Please make sure that you check the NDCQ condition.
   Grading scheme: 5 points in total: $\lambda = 0$ case finding $(0, 0)$ 0.5 point, finding $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ one point , check NDCQ 0.5 point. Finding (1,0) and (0,1) one point and check NDCQ at $(1, 0)$ 0.5 point, NDCQ at $(0, 1)$ 0.5 point, finding $\lambda = 1$ 0.5 point. Determie maximizer and maximum value 0.5 point.
   Solution:
   Case 1 : $\lambda = 0$
   We have $-2\mu x - 2y = 0, -2x - 2\mu y = 0$ and $x^2 + y^2 = 1$. It can be simplified as $\mu x + y = 0, x + \mu y = 0$ and $x^2 + y^2 = 1$. So $y = -\mu x$ and $x - \mu^2 x = 0$. Thus $x = 0$ or $\mu = \pm 1$. If $x = 0$ then $y = 0$. But $(0, 0)$ doesn't satisfy $x + y \geq 1$.
   If $\mu = \pm 1$ then $y = \pm x$. From $x^2 + y^2 = 1$, we have $x = \pm\frac{\sqrt{2}}{2}$. So $(x, y) = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}), (\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$), $(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}), (-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$. But only $(x, y) = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ satisfies $x+y \geq 1$. The constraint inequalities are $x + y \geq 1$. At $(x, y) = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$, we have $x + y = \sqrt{2} > 1$. So it satisfies the NDCQ. $f(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}) = 1 - 2 \cdot \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2} = 0$.
   Case 2 : $\lambda > 0$
   We have $x + y - 1 = 0$. Solving $x + y - 1 = 0$ and $x^2 + y^2 = 1$, we have $(x, y) = (0, 1)$ and $(1, 0)$.
   If $(x, y) = (1, 0)$ then $-2y+\lambda-2\mu x = 0$ and $-2x+\lambda-2\mu y = 0$ imply $\lambda-2\mu = 0$ and $-2+\lambda = 0$. Hence $\lambda = 2$ and $\mu = 1$. Hence $(1, 0)$ satisfies all conditions.
   If $(x, y) = (0, 1)$ then $-2y+\lambda-2\mu x = 0$ and $-2x+\lambda-2\mu y = 0$ imply $-2+\lambda = 0$ and $\lambda-2\mu = 0$. Hence $\lambda = 2$ and $\mu = 1$. Hence $(0, 1)$ satisfies all conditions.
   At $(0, 1)$ and $(0, 1)$, we have $g(x, y) = x + y = 1$ But $\nabla g = (1, 1) \neq 0$. So $(0, 1)$ and $(0, 1)$ satisfy NDCQ conditions. $f(1, 0) = 1$, $f(0, 1) = 1$. Mote that $f(1, 0) = f(0, 1) > f(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}) = 0$.
   Since $x^2 + y^2 = 1$ and $x + y \geq 1$ is closed and bounded, f is continuouss and it achieves its global maximum in the constraint set. So the maximizers are $(0, 1)$ and $(1, 0)$ and the global maximum is 1

4. It is required to maximize $f = x^2 - 2xy + 2y^2$ subject to $x + y \leq 1$, $x \geq 0, y \geq 0$.

(a) (1 pts) Write down the classical Lagrangian for this problem, $L(x, y, \lambda_1, \lambda_2, \lambda_3)$, where λ_1, λ_2 and λ_3 are the Lagrange multipliers.
Grading scheme: One point Solution: The constraint is $x + y \leq 1$, $-x \leq 0$ and $-y \leq 0$. So $L(x, y, \lambda_1, \lambda_2, \lambda_3) = x^2 - 2xy + 2y^2 - \lambda_1(x + y - 1) - \lambda_2(-x) - \lambda_3(-y) = x^2 - 2xy + 2y^2 - \lambda_1(x + y - 1) + \lambda_2 x + \lambda_3 y$.

(b) (2 pts) State the five equality and six inequality conditions required to determine the maximizers. ([You are not required to find the stationary points in this problem,)
Grading scheme: Each part 2/11 point.
Solution: We have $\frac{\partial L}{\partial x} = 2x - 2y - \lambda_1 + \lambda_2 = 0$, $\frac{\partial L}{\partial y} = -2x + 4y - \lambda_1 + \lambda_3 = 0$, $\lambda_1(x + y - 1) = 0, \lambda_2 x = 0, \lambda_3 y = 0$, $\lambda_1 \geq 0, \lambda_2 \geq 0$ and $\lambda_3 \geq 0$, $x + y \leq 1$, $x \geq 0$ and $y \geq 0$.